CURVES OF PURSUIT.

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Problem.—A point, ***B***, moves with a uniform velocity along a given straight line: another point, ***A***, moves continually toward ***B*** with a velocity also uniform. Required the locus of ***A***.

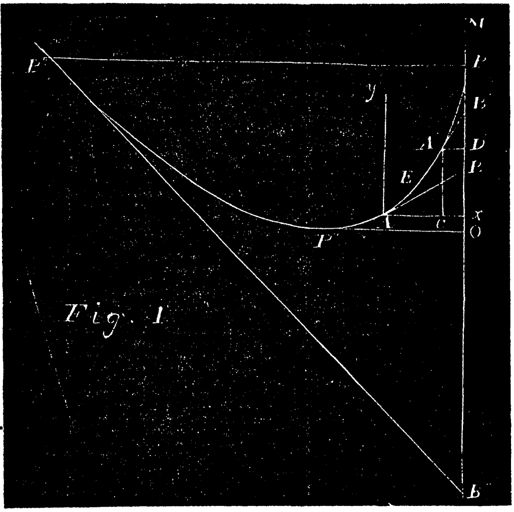


Fig. 1

Let MO, Fig. 1, be the given straight line, and A and E the positions of the points ***A*** and ***B***, respectively, at any instant. Suppose that while ***B*** moves from B to B', ***A*** moves from A to A', describing the curve AEA'. Then from the nature of the problem the right lines AB and $A'B'$ are tang' to the curve AEA' at A and A', respectively.

Take A as the origin of rectangular co-ordinates, of which AX and AY, respectively perpendicular and parallel to MO, are the axes. Draw $A'D$ perpendicular, and $A'C$ parallel to MO; and make $AC = x$, $A'C = y$, $A'B' = t$, $A'D = z$, $A'B'M = \varphi$, $DA'B' = \theta$, curve $AEA' = s$, $AB = T$, $AX = b$, and $ABM = \beta$. Of these quantities the last three relate to the relative initial positions of the points ***A*** and ***B***, and may be considered constant. Further, if $m =$ the ratio of the two given velocities, $BB' = ms$.

Since $B'D + DX = B'B + BX$, therefore

$$(b-x)\frac{dy}{dx} + y = ms + \text{constant}.$$

Or, differentiating,

$$(b-x)\frac{d^2y}{dx^2} = m\frac{ds}{dx}. \tag{1}$$

By substituting for $\frac{ds}{dx}$ its value, $\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}$, the above equation may be written

$$\frac{d\,\frac{dy}{dx}}{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}} = \frac{m\,dx}{b-x},$$

the integration of which gives

$$\log\left[\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right]=-\log(b-x)^m+\log C.$$

At the origin,

$$\frac{dy}{dx}=-\cot\beta;\text{ and }\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}=\operatorname{cosec}\beta.$$

Therefore $C=b^m(\operatorname{cosec}\beta-\cot\beta) = b^m\tan\frac{1}{2}\beta$; and consequently,

$$\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}=\frac{b^m\tan\frac{1}{2}\beta}{(b-x)^m}. \tag{2}$$

Hence we derive

$$\frac{dy}{dx}=\frac{b^m\tan\frac{1}{2}\beta}{2(b-x)^m}-\frac{(b-x)^m}{2b^m\tan\frac{1}{2}\beta}, \tag{3}$$

and

$$\frac{ds}{dx}=\frac{b^m\tan\frac{1}{2}\beta}{2(b-x)^m}+\frac{(b-x)^m}{2b^m\tan\frac{1}{2}\beta}. \tag{4}$$

In the above equations $b-x=z=$ the perpendicular from any point of the curve upon MO. Call that perpendicular which is tangent to the curve, a. ($P'O$, Fig. 1.) Then, from (3), we have, $a^m=b^m\tan\frac{1}{2}\beta$; or $b^m=a^m\times\cot\frac{1}{2}\beta$. Therefore, since the origin may be at any point of the curve, we have

$$(b-x)^m=z^m=a^m\cot\tfrac{1}{2}\varphi. \tag{5}$$

The integration of equations (3) and (4) gives different forms according as $m<1$, $m=1$, or $m>1$. We will discuss each case separately.

CASE 1. $m<1$.

Using a^m instead of $b^m\tan\frac{1}{2}\beta$, and integrating equation (3), we have, since $b=T\sin\beta$,

$$y=\frac{(b-x)^{1+m}}{2a^m(1+m)}-\frac{a^m(b-x)^{1-m}}{2(1-m)}+\frac{T(m-\cos\beta)}{1-m^2}. \tag{6}$$

The integration of equation (4) gives,

$$s=\frac{T(1-m\cos\beta)}{1-m^2}-\frac{(b-x)^{1+m}}{2a^m(1+m)}-\frac{a^m(b-x)^{1-m}}{2(1-m)}. \tag{7}$$

When A overtakes B (at P, Fig. 1) we shall have $x=b$; and the above equation reduces to

$$y=PX=\frac{T(m-\cos\beta)}{1-m^2};\text{ and }s=AEA'P=\frac{T(1-m\cos\beta)}{1-m^2}.$$

Equations (6) and (7) may be simplified by changing the origin. Thus if we change the origin to P', we shall have $a=b=T$, and $\beta=\frac{1}{2}\pi$. Therefore equation (6) reduces to

$$y=\frac{(a-x)^{1+m}}{2a^m(1+m)}-\frac{a^m(a-x)^{1-m}}{2(1-m)}+\frac{am}{1-m^2}, \tag{8}$$

and $$s = \frac{a}{1-m^2} - \frac{(a-x)^{1+m}}{2a^m(1+m)} - \frac{a^m(a-x)^{1-m}}{2(1-m)}. \tag{9}$$

Therefore $PO = \dfrac{am}{1-m^2}$; and $PEP' = \dfrac{a}{1-m^2}$; and consequently,

$$PEP' = (1\div m)PO.$$

Since $m < 1$, we may make $\cos\beta = m$, and thus cause the constant term in equation (6) to disappear. For this particular value of β, we find from equation (5), $b = a\left(\dfrac{1+m}{1-m}\right)^{\frac{1}{2m}}$, which call c. Therefore equation (6) becomes, in terms of c,

$$y = \frac{(c-x)^{1+m}}{2c^m(1-m^2)^{\frac{1}{2}}} - \frac{c^m(c-x)^{1-m}}{2(1-m^2)^{\frac{1}{2}}}. \tag{10}$$

Since $y = 0$, in the above equation, when $x = 0$, and when $x = c$, it is evident that this new axis of X, is perpendicular to OM at P where $\mathcal{A}$ overtakes $\mathcal{B}$; and that the curve cuts the axis of X at P'' at an angle whose sine is m. From equation (7) we have

$$s = T - \frac{(c-x)^{1+m}}{2c^m(1-m^2)^{\frac{1}{2}}} - \frac{c^m(c-x)^{1-m}}{2(1-m^2)^{\frac{1}{2}}}, \tag{11}$$

where $T = P''B''$. Therefore when $x = c$, we have $s = T$. That is, the curve $PP'P''$ is equal to the tangent $P''B''$.

An expression for t, that is, the distance between the points $\mathcal{A}$ and $\mathcal{B}$ at any instant, may easily be found. It is evident from the figure that

$$t = (b-x)\frac{ds}{dx} = z\frac{ds}{dx} = z\operatorname{cosec}\varphi = z\sec\theta. \tag{12}$$

But from equation (4)

$$\frac{ds}{dx} = \frac{a^{2m}+z^{2m}}{2a^m z^m}. \quad \therefore t = \frac{(a^{2m}+z^{2m})z^{1-m}}{2a^m}.$$

Another expression for t may be found as follows:— Differentiate the first of equations (12), and we have,

$$\frac{dt}{dx} = (b-x)\frac{d^2s}{dx^2} - \frac{ds}{dx};$$

but from equations (3) and (4) we easily deduce

$$(b-x)\frac{d^2s}{dx^2} = m\frac{dy}{dx}.$$

Therefore

$$\frac{dt}{dx} = m\frac{dy}{dx} - \frac{ds}{dx}. \tag{13}$$

$$\therefore t = my - s + T. \tag{14}$$

When $t = 0$, $s = my + T$. That is (Fig. 1), $AEP = AB + m \times PX$.

Radius of Curvature.

Designate the radius of curvature by R. Then, since generally,

$$R = \pm \frac{\frac{ds}{dx}}{\frac{d^2y}{dx^2}} \cdot \frac{ds^2}{dx^2};$$

if we use the plus sign and reduce by means of equation (1) we shall have,

$$R = \frac{b - x}{m} \cdot \frac{ds^2}{dx^2}. \qquad (15)$$

Therefore by equation (12)

$$R = \frac{t^2}{mz} = \frac{t \sec\theta}{m} = \frac{a \cot^{\frac{1}{}} \frac{1}{2}\varphi}{m \sin^2\varphi}.$$

Substituting for $\frac{ds}{dx}$, in equation (15), its value already found, we have,

$$R = \frac{(a^{2m} + z^{2m})^2 z^{1-2m}}{4ma^{2m}} \text{ or } \frac{(a^{2m} + z^{2m})^2}{4ma^{2m}z^{2m-1}},$$

according as $m <$ or $>$ than $\frac{1}{2}$. When $m = \frac{1}{2}$, we have the simple exp'n,

$$R = \frac{(a + z)^2}{2a}.$$

From the above expression it will be seen that the radius of curvature at P is zero when m is less than one half; $\frac{1}{2}a$ when m = one half; and infinite when m is greater than one half. The radius of curvature at P' is, in all cases, $\frac{a}{m}$; and at P'', $R = \frac{c}{m(1 - m^2)}$.

To determine whether R has a minimum, we differentiate equation (15), and obtain,

$$\frac{dR}{dx} = \left(2\frac{dy}{dx} - \frac{1}{m} \cdot \frac{ds}{dx}\right)\frac{ds}{dx} = \frac{2m \sin\theta - 1}{m \cos^2\theta}.$$

Therefore when $dR \div dx = 0$, we have $\sin\theta = (1 \div 2m)$.
The second differential coefficient is

$$\frac{d^2R}{dx^2} = \frac{2m}{b-x}\left(\frac{dy^2}{dx^2} + \frac{ds^2}{dx^2}\right) - \frac{2}{b - x} \cdot \frac{dy}{dx} \cdot \frac{ds}{dx}, \text{ which reduces to } \frac{2m}{b-x}$$

on the supposition that $dR \div dx = 0$. As this is positive, R is a minimum when $\sin\theta = \frac{1}{2m}$, or $z = a\left(\frac{2m - 1}{2m + 1}\right)^{\frac{1}{2m}}$.

If we designate by R_m the minimum value of R, we shall have

$$R_m = \frac{4am}{(2m + 1)^{\frac{2m+1}{2m}}(2m - 1)^{\frac{2m-1}{2m}}}.$$

We may also have by means of the preceding formulas, the following relations viz.:—

$$R_m = \frac{4mz}{4m^2-1} = 2z\tan\theta\sec\theta = 2t\tan\theta = \frac{2t}{\sqrt{(4m^2-1)}}.$$

In these formulas, z, t and θ refer, of course, to the point of maximum curvature.

If we represent m by the proper fraction $p \div q$, where p and q are prime to each other, there will be three varieties of the curve, viz.:—when p is odd and q even; when p is even and q odd; and when p and q are both odd.

Before continuing the discussion it will be convenient to change the origin to P, at which point $b = T = 0$; and $\beta = \pi$. Therefore equations (6) and (7) become, changing the signs of x and s,

$$y = \frac{a^m x^{1-m}}{2(1-m)} - \frac{x^{1+m}}{2a^m(1+m)}; \qquad s = \frac{a^m x^{1-m}}{2(1-m)} + \frac{x^{1+m}}{2a^m(1+m)}$$

Figure 2 represents the first variety, where $m = \frac{1}{2}$. We have

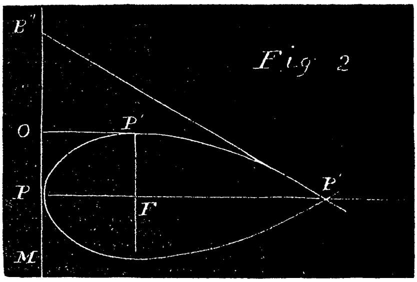
$$y = \pm\frac{(3a-x)\sqrt{x}}{3\sqrt{a}}; \qquad \text{and } s = \frac{(3a+x)\sqrt{x}}{3\sqrt{a}}.$$

Also $t = \pm\dfrac{a+x}{2}\left(\dfrac{x}{a}\right)^{\frac{1}{2}}$,

and $\quad R = \dfrac{(a+x)^2}{2a} = \dfrac{2t^2}{x} = 2t\sec\theta.$

The curve extends from $x = 0$ to $x = +\infty$, and is symmetrical with respect to the axis of x.

There is a multiple point at P'' where the two branches intersect at an angle of 60°. The following values are easily deduced:—



Arc $PP'P'' = B''P'' = a\sqrt{12}$. Arc $PP' = 2\times OP = \frac{4}{3}a$. $PP'' = 3\times OP'$. The values of R at P, P', and P'', are respectively, $\frac{1}{2}a$, $2a$ and $8a$—three quantities in geometrical progression. It is, perhaps worth mentioning that if the arc $PP'P''$ be revolved about PP'' as an axis, the volume generated will be $\frac{9}{16}$ of the circumscribing cylinder.

Figure 3 is a type of the second variety. In its construction, m was made equal to $\frac{4}{5}$. Its equation is

$$y = \frac{5x^{\frac{1}{5}}}{18a^{\frac{4}{5}}}\left[9a^{\frac{8}{5}} - x^{\frac{8}{5}}\right].$$

In all of these examples, by changing the sign of the second term within the brackets, of the equation of the curve, we have an expression for s; i. e., the length of the curve measured from P to the point whose abscissa is x

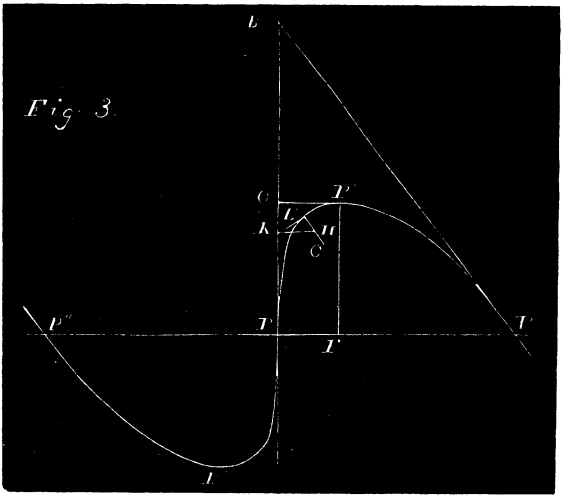


Fig. 3.

The curve extends from $x = + \infty$ to $x = - \infty$; and the parts on opposite sides of the axis of y are similar. The maximum curvature is at E, where

$$x = \left(\frac{3}{13}\right)^{\frac{5}{8}} a.$$

The minimum radius of curvature may be constructed as follows: At E draw the tangent EK and the normal EC. Draw KH perpendicular to the axis of y, and take $HC = EH$. Then is EC the radius of curvature at E. The radius of curvature at P is infinite, as it is in all these curves when $m > \frac{1}{2}$.

The third variety is illustrated by Fig. 4, where $m = \frac{1}{3}$. Its eq'n is

$$y = \frac{3x^{\frac{2}{3}}}{8a^{\frac{1}{3}}} \left[2a^{\frac{2}{3}} - x^{\frac{2}{3}} \right].$$

The curve consists of two similar branches extending to infinity on either side of the axis of y.

The radius of curvature at the origin is zero.

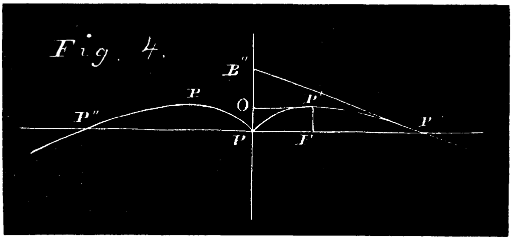


Fig. 4.

CASE 2. $m = 1$.

The integration of equations (3) and (4) gives, when $m = 1$,

$$y = \frac{a}{2} \log \left(\frac{b}{b-x}\right) + \frac{(b-x)^2}{4a} - \frac{b^2}{4a},$$

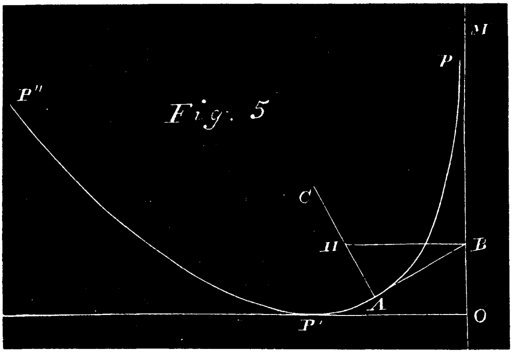
$$s = \frac{a}{2} \log \left(\frac{b}{b-x}\right) - \frac{(b-x)^2}{4a} + \frac{b^2}{4a}.$$

As before, the origin may be at any point of the curve determined by the value of b. If the origin be at P', Fig. 5, we shall have $b = a$ and the above equations reduce to

$$y = \frac{a}{2}\log\left(\frac{a}{a-x}\right) + \frac{x^2}{4a} - \frac{x}{2} = \frac{a}{2}\log\left(\frac{a}{z}\right) + \frac{z^2}{4a} - \frac{a}{4},$$

$$s = \frac{a}{2}\log\left(\frac{a}{a-x}\right) - \frac{x^2}{4a} + \frac{x}{2} = \frac{a}{2}\log\left(\frac{a}{z}\right) - \frac{z^2}{4a} + \frac{a}{4}.$$

When $x = a$, y and s are both infinite; but from the above equations we have, when $x = a$, $s - y = \frac{1}{2}a$. That is, if the curved line $P'AP$, which meets the straight line OM at infinity were straightened or developed along OM, it would reach below O by a distance equal to $\frac{1}{2}a$.



The following values of t are easily deduced, viz :—

$$t = z\sec\theta = \frac{a}{1+\sin\theta} = \frac{dx}{d\theta} = a + y - s = \frac{z^2}{2a} + \frac{a}{2} = \sqrt{(Rz)}.$$

Therefore when $z = 0$, $t = \frac{1}{2}a$; which is the distance between the points A and B, at infinity.

The following are some of the expressions for R, viz. :—

$$R = \frac{t^2}{z} = t\sec\theta = \frac{(a^2+z^2)^2}{4a^2z}.$$

When R is a minimum $\theta = 30°$, $z = a \div \sqrt{3}$ and $t = 2a \div 3$; and the minimum value of R is $R_m = 4a \div 3\sqrt{3}$. A, Fig. 5, is the point of maximum curvature, and $AC = 2 \times AH$ is the radius of curvature. The radius of carvature at any point of the curve whose coordinates are given may be easily constructed as follows: Let A be any point of the curve. Then by means of its coordinates, the tangent AB and normal AC may be drawn. From B draw BH parallel to OP'. Then is BH equal to the radius of curvature at the given point. At P', $R = a$.

The area between the curve $P'P$ and its asymptote is $\frac{1}{3}a^2$; and the volume generated by revolving $P'P$ about $P'O$ is $\frac{14}{45}\pi a^3$; which is therefore the volume of a disc whose base is a circle of infinite radius and whose altitude is a.

If the curve PP' revolve about its asymptote, the surface generated by the curve $= \frac{4}{3}\pi a^2$, and the volume $= \frac{1}{8}\pi a^3$.

Case 3. $m > 1$.

When $m > 1$, the integration of equations (3) and (4) gives, placing the origin at P', Fig. 6.

$$y = \frac{a^m}{2(m-1)(a-x)^{m-1}} + \frac{(a-x)^{m+1}}{2a^m(m+1)} - \frac{am}{m^2-1},$$

$$s = \frac{a^m}{2(m-1)(a-x)^{m-1}} - \frac{(a-x)^{m+1}}{2a^m(m+1)} - \frac{a}{m^2-1}.$$

Although when m is finite, s and y are both infinite when $x = a$, yet their difference, as in case 2, is finite. The value of $s - y$ is, in all cases, equal to $a \div (m+1)$.—We have already found

$$t = \frac{a^{2m} + z^{2m}}{2a^m z^{m-1}},$$

which is infinite when $z = 0$. To determine whether t has a minimum value, we make the second member of equation (13) equal to zero, and find $\sin\theta = (1 \div m)$. This substituted in the second diff. coeff't reduces it to

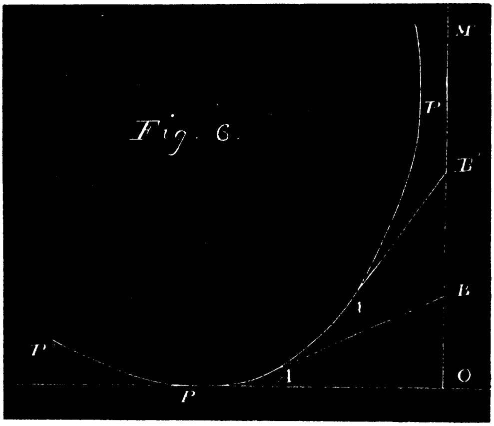
$$\frac{m}{a}\left[(m+1)^{\frac{m+1}{2m}}(m-1)^{\frac{m-1}{2m}}\right];$$

and this is positive, so long as $m > 1$, for positive roots; and those are the only ones considered in this discussion. The preceding formulæ give, when $\sin\theta = (1 \div m)$,

$$z = a\left(\frac{m-1}{m+1}\right)^{\frac{1}{2m}};$$

and thence we deduce for t_m the following value, viz. :—

$$t_m = \frac{am}{(m+1)^{\frac{m+1}{2m}}(m-1)^{\frac{m-1}{2m}}}.$$



We have already found, when $\sin\theta = 1 \div (2m)$,

$$R_m = \frac{4am}{(2m+1)^{\frac{2m+1}{2m}}(2m-1)^{\frac{2m-1}{2m}}},$$

and

$$z = a\left(\frac{2m-1}{2m+1}\right)^{\frac{1}{2m}}.$$

In Fig. 6, where $m = \frac{5}{4}$, A is the point of maximum curvature, and $A'B'$ the minimum distance between the points.

For all values of $m > 1$, A lies between A' and P'; and as m is increased A and A' approach still nearer to P'.